Worksheet 9, April 25, 2025

1 Deriving a new quadrature rule

Given $f : [0,1] \to \mathbb{R}$, you want to derive a new quadrature rule that does uses not only function values, but also gradient values:

$$\int_0^1 f(x)\,dx \approx \alpha_0 f(0) + \alpha_1 f'(0) + \alpha_2 f(1). \tag{1}$$

Q1 First, find polynomials $J_0, J_1, J_2 \in \mathcal{P}_2$, with the following properties:

$$\begin{aligned} J_0(0) &= 1, \quad J_0'(0) = 0, \quad J_0(1) = 0 \\ J_1(0) &= 0, \quad J_1'(0) = 1, \quad J_1(1) = 0 \\ J_2(0) &= 0, \quad J_2'(0) = 0, \quad J_2(1) = 1. \end{aligned}$$

(*Hint:* For each J_i, make an ansatz for a quadratic polynomial using the monomial basis.)

Given f, you can now define a polynomial approximation $p \in \mathcal{P}_2$ via

$$p(x) = f(0)J_0(x) + f'(0)J_1(x) + f(1)J_2(x). \tag{2}$$

The polynomial p is an approximation to f in the sense that $p(0) = f(0)$, $p'(0) = f'(0)$ and $p(1) = f(1)$.

Q2 Use the polynomial p derived in (2) and the same method used to derive the Newton-Cotes quadrature rules, to find the coefficients α_0, α_1 and α_2 in (1).

Q3 Use your new quadrature rule to approximate $\int_0^1 \exp(2x)\sin^2(x)\,dx$, and also compare with Simpson's rule. The exact value of this integral is $1.2668\ldots$.

2 Trapezoidal rule for smooth periodic functions

We investigate how the (composite) trapezoidal rule performs for smooth, periodic functions. Consider integrating the smooth, periodic function $f(x) = e^{\sin x}$ over a single period. The exact value of the integral is

$$I(f) = \int_0^{2\pi} e^{\sin x}\,dx = 7.95492652101284527\ldots.$$

Q1 Write down the composite trapezoidal rule $T_N(f)$ on equispaced nodes $0 = x_0 \le \cdots \le x_N = 2\pi$ for estimating the value of this integral.

Q2 Simplify your expression for $T_N(f)$ using the periodicity of f. How many function evaluations do you have for a given N?

Q3 Show that $T_N(f)$ is equivalent to both a left-endpoint Riemann sum and a right-endpoint Riemann sum approximation to $I(f)$.

Q4 Compute $T_N(f)$ for various progressively larger N. Plot the quadrature errors against N on (i) a log-log plot, and (ii) a `semilogy` plot. What is the order of accuracy of the trapezoidal rule for smooth, periodic functions?