Worksheet 8, April 18, 2025

1 Lagrange interpolation polynomial

Recall that the Lagrange interpolation of a function f at the points $x_0, x_1, ..., x_n$ has the form

$$p_n(x) = \sum_{j=0}^{n} L_j(x) f(x_j)$$

where

$$L_j(x_i) = \prod_{i=0, i\neq j}^{n} \frac{x - x_i}{x_j - x_i}$$

Q1 True or False? For the nodes $x_0 = 0, x_1 = 1, x_2 = 2$, the Lagrange interpolation polynomial $L_0(x)$ is $-x^2 + 1$.

Q2 Given the distinct points x_i for $i = 0, 1, ..., n + 1$ and the points y_i, $i = 0, 1, ..., n + 1$, let $q(x)$ be the Lagrange polynomial of degree n for the set of points $\{(x_i, y_i) : i = 0, 1, ..., n\}$. Meanwhile, let $r(x)$ be the Lagrange polynomial of degree n for the points $\{(x_i, y_i) : i = 1, 2, ..., n + 1\}$. Define

$$p(x) = \frac{(x - x_0)r(x) - (x - x_{n+1})q(x)}{x_{n+1} - x_0}$$

Show that p is the Lagrange polynomial of degree $n + 1$ for the points $\{(x_i, y_i) : i = 0, 1, ..., n + 1\}$.

2 Hermite interpolation polynomial

Recall that the Hermite interpolation of a function f at the points $x_0, x_1, ..., x_n$ has the form

$$p_{2n+1}(x) = \sum_{j=0}^{n} H_j(x) f(x_j) + \sum_{j=0}^{n} K_j(x) f'(x_j)$$

with error

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n + 2)!} \left[\pi_{n+1}(x)\right]^2 \tag{1}$$

where $\pi_{n+1}(x) = \prod_{j=0}^{n}(x - x_j)$.

2.1

Q1 Construct the Hermite interpolation polynomial of degree 3 (i.e., $p_3(x)$) for the function $f(x) = x^5$ using the points $x_0 = 0$ and $x_1 = a$.

Q2 Verify (1) by direct calculation, showing that in this case ξ is unique and has the value $\xi = (x + 2a)/5$.

2.2

Show that the polynomial

$$-\frac{1}{\pi}x^2 + x$$

is the Hermite interpolation polynomial of $f(x) := \sin(x)$ based on the nodes $x_0 = 0$, $x_1 = \pi$.

2.3

True or False? The Hermite interpolation with 3 distinct nodes is exact for polynomials of degree 6. If False, how many nodes would made the interpolation exact?