Worksheet 7, April 11, 2025

1 Eigenvectors as stationary points of Rayleigh quotient

For H a real symmetric matrix, we define Rayleigh quotient as a function $\mathbb{R}^n \to \mathbb{R}$:

$$R(x) = \frac{x^\top H x}{x^\top x} = \frac{q(x)}{p(x)}.$$

Q1 Find gradient of $R(x)$ at v. In particular, take $t \in \mathbb{R}$ and calculate

$$\frac{d}{dt} R(v + ty)\bigg|_{t=0}$$

for all $y \in \mathbb{R}^n$. We can get the gradient by picking $y = e_i$.

Q2 Using the above calculation, show that v is a stationary point (i.e.: $\nabla R(v) = 0$) of the Rayleigh quotient if and only if it is an eigenvector of H.

2 QR decomposition via Householder

Construct the QR factorization of the following matrix using Householder reflectors:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}.$$

Use the factorization to determine $|\det(A)|$.

3 Rayleigh Quotient Iteration

The Rayleigh Quotient is a method for obtaining an eigenvalue estimate from an eigenvector estimate, while Inverse Iteration is a method for computing an eigenvector estimate from an eigenvalue one. The idea of the *Rayleigh Quotient Iteration* is to combine these methods to continually improve the eigenvalue estimate and increase the rate of convergence of inverse iteration at every step. The algorithm (as in Trefethen and Bau 1997) is given by:

Algorithm 27.3. Rayleigh Quotient Iteration

$v^{(0)}$ = some vector with $\|v^{(0)}\| = 1$
$\lambda^{(0)} = (v^{(0)})^T A v^{(0)}$ = corresponding Rayleigh quotient
for $k = 1, 2, \ldots$
  Solve $(A - \lambda^{(k-1)} I) w = v^{(k-1)}$ for w — apply $(A - \lambda^{(k-1)} I)^{-1}$
  $v^{(k)} = w / \|w\|$ — normalize
  $\lambda^{(k)} = (v^{(k)})^T A v^{(k)}$ — Rayleigh quotient

Q1 Implement the Rayleigh Quotient Iteration

Q2 What is the order of convergence of the eigenvalue estimate ($\lambda^{(k)}$) and eigenvector estimate ($v^{(k)}$)? Can you explain how this relates to the convergence rate of inverse iteration?