Worksheet 6, April 4, 2025

1 Proofs regarding eigenvalues

Q1 Prove that if X is nonsingular, then A and $X^{-1}AX$ (a *similarity transformation*) have the same characteristic polynomial, eigenvalues, and algebraic multiplicities.

Q2 Prove that a symmetric matrix A has real eigenvalues

2 Gershgorin disks

Consider the matrix

$$A = \begin{bmatrix} -6 & 2 & 0.3 & 0 & -0.7 \\ 2 & -4 & 0.1 & 0.05 & 0 \\ 0.3 & 0.1 & 2 & 0.1 & 0.1 \\ 0 & 0.05 & 0.1 & 4 & 0 \\ -0.7 & 0 & 0.1 & 0 & 6 \end{bmatrix}$$

and recall the definition of the Gershgorin disks:

$$D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}.$$

Q1 Argue that all eigenvalues of A are real.

Q2 What are the Gershgorin disks for A? Use them to give a set, $D \subset \mathbb{R}$, that contains all eigenvalues of A.

Q3 Can you conclude that the eigenvalue with the largest absolute value is simple? What about the largest positive eigenvalue?

Q4 Argue that A is invertible. Conclude that all diagonally dominant matrices are invertible.

Q5 True or False? Let $A \in \mathbb{R}^{n \times n}$ and D_i, $i = 1, 2, \ldots, n$, be the Gerschgorin disks of A. If $0 \in \bigcup_{i=1}^{n} D_i$ then A is singular. Prove if true, provide counter-example if false.

Consider now the matrix

$$B = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{bmatrix}$$

where $|\epsilon| < 1$.

Q6 Give estimates for the eigenvalues of B.

3 Computing Eigenvalues via the Power Iteration

Consider the matrix

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}$$

which has eigenvalues and eigenvectors

$$\lambda_1 = 0,\ \boldsymbol{v}_1 = \begin{bmatrix} 0.41 \\ -0.82 \\ 0.41 \end{bmatrix}, \quad \lambda_2 = -6,\ \boldsymbol{v}_2 = \begin{bmatrix} 0.71 \\ 0.0 \\ -0.71 \end{bmatrix}, \quad \lambda_3 = 3,\ \boldsymbol{v}_3 = \begin{bmatrix} -0.58 \\ -0.58 \\ -0.58 \end{bmatrix}.$$

Q1 Calculate the first iterate of the power method when $\boldsymbol{x}_0 = (0, 1, 1)^T$.

Q2 Which eigenvalue direction will the sequence defined in **Q1** converge to?

Q3 Give an initialization vector such that the power method does *not* converge to the direction of the largest (in absolute value) eigenvalue.

Q4 Write a simple program implementing the power method for the matrix A.

(a) Use the Rayleigh quotient to calculate estimates of the eigenvalues for each iteration.

(b) What is the theoretical order of convergence of the eigenvector estimates and the eigenvalue estimates? What is the speed of convergence? Does your implementation match?