Worksheet 5, March 7, 2025

1 Condition numbers and pivoted LU

Q1 Suppose we want to solve a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ for some $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$, finding unknown $\boldsymbol{x} \in \mathbb{R}^n$. If we perturb the vector $\boldsymbol{b}$ by a relative perturbation $\|\delta\boldsymbol{b}\|_2 / \|\boldsymbol{b}\|_2$, what can we expect regarding the relative error $\|\delta\boldsymbol{x}\|_2 / \|\boldsymbol{x}\|_2$ of the solution $\boldsymbol{x}$?

Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ for

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Q2 Solve for $\boldsymbol{x}$

Q3 What are $\kappa_2(A)$ and $\kappa_\infty(A)$?

Q4 Compute $\boldsymbol{x}$ if instead you add perturbations $\Delta\boldsymbol{b} = \left[10^{-3}, 0\right]^T$ or $\Delta\boldsymbol{b} = \left[0, 10^{-3}\right]^T$ to the right-hand-side. What do you notice? Do the relative errors in $\boldsymbol{x}$ and $\boldsymbol{b}$ agree with your answer in Q1?

Q5 Find the LU decomposition, with and without pivoting, of a matrix B given by

$$\boldsymbol{B} = \begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix}$$

Notice that $\boldsymbol{B}$ is well-conditioned. Are both LU decompositions (with and without pivoting) also well-conditioned?

2 Counting flops

Consider the pseudo code snippet below. Give the operation/flop count **exactly**.

```
z=0
for i in {5,...,n}
  for j in {1,2,3,4}
    for k in {1,...,n}
      z += (i-j)× k
    end
  end
end
```

3 Least squares and population models

Suppose you visit a lake where an invasive species of fish was recently introduced. You manage to record the population count of these fish, gathering measurements between days 30 and 70 since the invasive species was found:

Time (days)	30	32.5	39	52	61	61	68	69.5
Population count	40	50	69.5	152.3	220.9	235	315	373.2

On day 70, you happen to run into a person who claims to be an expert in population dynamics. They claim that the fish population is "obviously" described by

$$P(t) = a \exp\left(bt + ct^{1/2}\right) \qquad \text{Model 1} \tag{1}$$

with the right choice of parameters a, b, and c.

Q1 Write an equivalent equation to (1) for which the method of least squares applies. Find your matrix A and vector **b** to solve the problem

$$\min_{x \in \mathbb{R}^3} \|A\boldsymbol{x} - \boldsymbol{b}\|_2^2$$

Q2 Use QR factorization to solve this problem for your parameters. Plot the data points and model curve. Does this model look like a good fit? What is the error?

Q3 Since you do not trust this alleged expert in population dynamics, you decide to try two other models:

$$P(t) = a \exp(bt) \qquad \text{Model 2} \tag{2}$$

$$P(t) = a + bt + ct^2 \qquad \text{Model 3} \tag{3}$$

Use the method of least squares to find the best parameters for these alternative models, and plot them against your results from Model 1. How do they compare, qualitatively and quantitatively with the error?

Q4 Suppose you were interested in predicting how many invasive fish will be in the lake at day 100, choosing the "best" of Models 1, 2, and 3. Is this a "good idea"?