Worksheet 2, February 14, 2025

1 Complexity of computing matrix norms

1.1

Recall the formulas for the 1- and ∞-norm. Assume that taking absolute values $|x|$ and determining a maximum $\max_i x_i$ do not contribute to computational cost.

Q1 How many *flops* (floating point operations) are needed to calculate $\|A\|_1$ and $\|A\|_\infty$ for $A \in \mathbb{R}^{n\times n}$?

Q2 For large n, by what factor will the calculation time increase when you double the matrix size?

Q3 Implement a simple code that calculates $\|A\|_1$ and $\|A\|_\infty$ for a matrix of any size $n \geq 1$. Do this without using loops (many commands like `np.sum` can be applied to vectors, and are much faster).

Q4 For $n_k = 2^k$ for $k = 7, ..., 14$, time how long your code takes to calculate $\|A\|_1$ and $\|A\|_\infty$ for a matrix $A \in \mathbb{R}^{n_k \times n_k}$ with random entries and report the results.

Q5 Can you confirm the estimate from Q1? Demonstrate this in a plot.

2 Positive definite matrices

A matrix $A \in \mathbb{R}^n$ is positive definite if

$$x^T A x > 0 \qquad \forall \text{ nonzero } x$$

Q1 Consider a 2×2 matrix

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Derive the conditions under which the matrix A is positive definite.

Q2 Use your answer from Q1 to demonstrate that the determinant of a positive definite matrix $A \in \mathbb{R}^{2\times 2}$ is positive.

Q3 What can you conclude about the eigenvalues of a positive definite matrix $A \in \mathbb{R}^{2\times 2}$?

Q4 Show that for (any size) symmetric positive definite matrices $A \in \mathbb{R}^{n\times n}$, the 2-norm condition number can also be computed as the ratio between the largest and the smallest eigenvalue of A, i.e.: $\kappa_2(A) = \lambda_{\max}/\lambda_{\min}$.

3 Steffensen's Method

The goal of this task is to analyze Steffensen's Method for finding the root ξ of a function $f : \mathbb{R} \to \mathbb{R}$, defined by the iteration

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)}, \qquad k = 1, 2, \dots. \tag{1}$$

Q1 Explain how this iteration relates to Newton's method and the Secant method.

Q2 Show that the sequence $\{x_k\}_{k=0}^{\infty}$ converges quadratically if x_0 is sufficiently close to the solution.

(a) Step 1: Rewrite (1) by expanding $f(x_k + f(x_k))$ around x_k (to order $f(x_k)^2$).

(b) Step 2: Consider the Taylor series of $f(\xi)$ around x_k:

$$0 = f(\xi) = f(x_k) + f'(x_k)(\xi - x_k) + \frac{1}{2} f''(z_k)(\xi - x_k)^2$$

where z_k is some value between x_k and ξ. Use this in combination with part (a) to write an expression for $\xi - x_{k+1}$ in terms of $\xi - x_k$.

(c) Compute the limit

$$\lim_{k \to \infty} \frac{|\xi - x_{k+1}|}{|\xi - x_k|^2}$$

Q3 Compare the computational cost of Steffensen's method to that of Newton's method and the secant method. When would you use which of the methods?

Q4 If there is time, try implementing Steffensen's method and verify the convergence is as you expect.