Worksheet 2, February 21, 2025

1 Frobenius norm of a rank one matrix

Recall, from your homework, that the formula for the Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$ is given by:

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \tag{1}$$

Suppose that A is the outer product uv^T (i.e. rank one). Show that $\|A\|_F = \|u\|_F \|v\|_F$. (Note: for a vector, the Frobenius norm is equivalently the L^2 norm).

2 Schur complement

Consider a matrix $M \in \mathbb{R}^{(m+n)\times(m+n)}$, which we rewrite as a *block matrix*:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times n}$. We shall assume that M and all its leading submatrices are non-singular.

Q1 Verify the formula

$$\begin{bmatrix} I & \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ & D - CA^{-1}B \end{bmatrix}$$

for "elimination" of the block C. The matrix $D - CA^{-1}B$ is known as the *Schur complement* of A in M.

Q2 Explain the above decomposition as a form of "block LU".

3 Solving $Ax = b$ and LU factorization

Let's compute the LU-factorization of $A := \begin{bmatrix} 3 & 3 & 0 \\ 6 & 4 & 7 \\ -6 & -8 & 9 \end{bmatrix}$ using the following direct approach to find L and U:

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = A.$$

Q1 By multiplying appropriate rows and columns, find the entries of L and U in the following order: $u_{11}, u_{12}, u_{13}, l_{21}, l_{31}, u_{22}, u_{23}, l_{32}, u_{33}$.

Q2 Use the LU factorization to solve the linear system $Ax = b$ with $b = [1, 0, 0]^\top$ using one forward and one backward substitution.

Q3 Use the LU factorization to compute the determinant of A. Recall that for two matrices of appropriate sizes, $\det(AB) = \det(A)\det(B)$.

Q4 In the matrix A defined above, replace the $(2,2)$-entry by 6. What is the rank of A after this modification? Are you able to compute the LU factorization of A as before? How might you "fix" the problem that arises?

4 LU factorization and sparsity

For this question, you will use a matrix A provided on Brightspace found in the files `matrixA.mat` (MATLAB) or `matrixA.npy` (python). Note that in MATLAB it is a *sparse* matrix.

Q1 Use `spy` to display the structure of A. Can you quantify the sparsity?

Q2 Find the inverse of A through `inv(A)`/`scipy.linalg.inv(A)`, and use `spy` to display the structure. What do you notice?

Q3 We will use built-in MATLAB/python functions to find the LU factorization of A. Either: `[L,U] = lu(A)`/`L, U = scipy.linalg.lu(A, permute_l=True)`.
What do the structures of L and U look like? How do they compare to A? Do you see *fill in*, where what was once a zero entry in A is now nonzero?

Q4 Now, load in the larger matrix A. Generate a random vector b of appropriate size to consider the problem $Ax = b$. Compare the compute-times of the following:

- Computing A^{-1}
- Computing $x = A^{-1}b$
- Computing L and U
- Computing x via forward and back substitution

Given what you find, how would you solve the problem $Ax = b$? Explain your answer.